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International Journal of Heat and Mass Transfer 49 (2006) 341–348

International Journal of HEAT and MASS TRANSFER

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Transient analytical solution to heat conduction in composite circular cylinder

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Received 14 April 2005
Available online 25 August 2005

Abstract

An analytical method leading to the solution of transient temperature filed in multi-dimensional composite circular cylinder is presented. The boundary condition is described as time-dependent temperature change. For such heat conduction problem, nearly all the published works need numerical schemes in computing eigenvalues or residues. In this paper, the proposed method involves no such numerical work. Application of 'separation of variables' is novel. The developed method represents an extension of the analytical approach derived for solving heat conduction in composite slab in Cartesian coordinates. Close-formed solution is provided and its agreement with numerical result is good which demonstrates a good accuracy of the developed solution form.

Keywords: Multi-dimension; Composite circular cylinder; Heat conduction; Analytical method; Close-formed solution

1. Introduction

Composite cylindrical shells are broadly used in contemporary, nuclear, aerospace, water resources and many other industries. Classical heat conduction in shell structures is obviously very important in studying their thermal load and deformation. There exist a great amount of numerical programs for evaluating heat conduction performances in such structures. Nevertheless, there are indeed many good reasons for deriving analytical solutions such as validating numerical models and analysing basic physical processes.

Theoretically, analytical methods in cylindrical geometry are completely analogous to those applied to Cartesian coordinates. In most analytical studies, one-dimensional geometry is widely investigated. The difficulties in multi-dimensional cases are much more profound. Commonly applied techniques are finite integral transform which is often employed to single layer material, Green function, orthogonal expansion and Laplace transform [1]. In Cartesian coordinates, examples of application of these techniques are Salt [2,3], Mikhailov and Özisik [4] (orthogonal expansion technique) as well as Haji-Sheikh and Beck [5] (Green function). In cylindrical coordinates, example works are Abdul Azeez and Vakakis [6] (integral transform) and Milosevic and Raynaud [7]. Numerical iterations on searching eigenvalues were needed in all of the above-cited papers. For multi-dimensional problems, associated eigenvalues

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doi:10.1016/j.ijheatmasstransfer.2005.06.019

Nomenclature

b	resultant coefficient when applying 'separation of variables'	T	temperature
F, G	functions defined in Eq. (3.15a)	U	homogenised temperature $= T - T_\infty$
h	intermediate variable defined in Eq. (3.13)	u, X	variable-separated temperatures $U = uX$
j	composite layer identifier	x	space coordinate
k	diffusivity		
l	layer thickness for circular cylinder	*Greek symbols*	
m	index number	α	convective and radiative heat transfer coefficient
n	layer number	η	intermediate variable defined in Eq. (3.13)
q	intermediate variable defined in Eq. (3.13)	φ	phase
R	constructed new variable defined in Eq. (3.11)	λ	thermal conductivity
r	space coordinate	ω	period
t	time	ξ	intermediate variable defined in Eq. (3.13)

may become imaginary which produce instability to the numerical iteration [5]. Such numerical instability occurs in applying the technique of Laplace transform also, as calculations often yield residue computation on numerically searching for the roots of hyperbolic equations [8].

In conclusion, eigenvalue and residue computations have always posed challenge to analytical methods on solving heat conduction in composite structures. Recently, a novel analytical method was developed to tackle transient heat problems for one-dimensional and multi-dimensional composite slab in Cartesian coordinates subject to time-dependent temperature changes [8,9]. It is free of numerical calculation. The objective of this paper is to extend the method to tackling multi-dimensional heat conduction in composite cylindrical shells. As calculation methods and results exhibit differently for varied types of cylindrical geometry, composite circular cylinder is considered here.

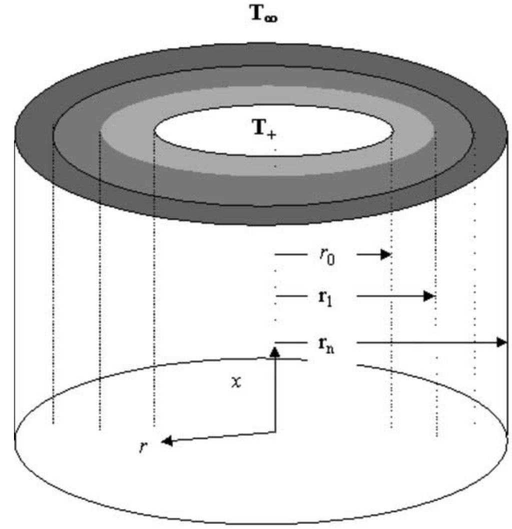

Fig. 1. Schematic of the composite circular cylinder.

2. Mathematical model

2.1. Model equations

Let n-layer composite circular cylinder be in cylindrical form in x- and r-directions as illustrated in Fig. 1. The layers are in r-direction and formed with different materials characterised by constant conductivity, diffusivity and thickness which are presented as λ_j, k_j and l_j, $j = 1, \ldots, n$. An ideal contact between layers is assumed.

Denote $r_0 = l_0$ and $r_j = l_0 + \cdots + l_j$, $j = 1, \ldots, n$. So the layer boundaries in r-direction are $r_0, r_1, \cdots, r_n$. The basic heat conduction equation in terms of temperature $T_j(t, r, x)$ in the cylindrical coordinates becomes

$$k_j\left(\frac{\partial^2 T_j}{\partial r^2} + \frac{1}{r}\frac{\partial T_j}{\partial r}\right) + k_j\frac{\partial^2 T_j}{\partial x^2} = \frac{\partial T_j}{\partial t},$$
$$r \in [r_{j-1}, r_j], \quad x \in [0,1], \quad j = 1, \ldots, n \tag{2.1a}$$

with boundary conditions

$$-\lambda_1\frac{\partial T_1}{\partial r}(t, r_0, x) = -\alpha_+(T_1(t, r_0, x) - T_+(t)),$$
$$x \in [0,1] \tag{2.1b}$$
$$T_j(t, r_j, x) = T_{j+1}(t, r_j, x),$$
$$x \in [0,1], \quad j = 1, \ldots, n-1 \tag{2.1c}$$

$$-\lambda_j\frac{\partial T_j}{\partial r}(t,r_j,x)=-\lambda_{j+1}\frac{\partial T_{j+1}}{\partial r}(t,r_j,x),\quad x\in[0,1],\ \ j=1,\dots,n-1 \tag{2.1d}$$

$$-\lambda_n\frac{\partial T_n}{\partial r}(t,r_n,x)=-\alpha_\infty(T_\infty(t)-T_n(t,r_n,x)), \tag{2.1e}$$

$$-\lambda_j\frac{\partial T_j}{\partial x}(t,r,0)=-\alpha_0(T_j(t,r,0)-T_\infty(t)),\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{2.1f}$$

$$-\lambda_j\frac{\partial T_n}{\partial x}(t,r,1)=-\alpha_1(T_\infty(t)-T_j(t,r,1)),\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{2.1g}$$

$$T_j(0,r,x)=0,\quad r\in[r_{j-1},\ r_j],\ \ x\in[0,1],j=1,\dots,n \tag{2.1h}$$

Here, without losing generality, it is assumed that the composite thickness in x-direction is 1, the initial temperature is zero and the surface heat transfer coefficients for ambient boundaries are α_+, α_∞, α_0 and α_1. Boundary temperatures are given as time-dependent $T_+(t)$ and $T_\infty(t)$ (see Fig. 1).

2.2. Further statement of the problem

For simplicity, we firstly assume simple boundary temperatures as $T_+(t)=\cos(\omega_+t+\varphi_+)$, $T_\infty(t)=\cos(\omega_\infty t+\varphi_\infty)$. Furthermore, for calculational convenience, solution will be given according to the complex form of the boundary temperature, namely

$$T_+(t)=\mathrm{e}^{\mathrm{i}\omega_+t+\mathrm{i}\varphi_+} \tag{2.2a}$$

$$T_\infty(t)=\mathrm{e}^{\mathrm{i}\omega_\infty t+\mathrm{i}\varphi_\infty} \tag{2.2b}$$

Clearly, the equation solution is the real part of the sought-after solution. If there is no danger of confusion we shall keep the same notations for the complex form of the boundary temperatures. More general time-dependent boundaries will be discussed later.

In general study contexts, it has been agreed that the boundary condition of the third kind can produce mathematical incompatibilities in the direction parallel to the layers (e.g. [4] and [8]). Hence, only the first and the second kind boundaries in r-direction are considered: α_0 and α_1 take the values of 0 or ∞ (first and second kinds), which leads to four boundary conditions in r-direction, namely

$$x\text{-boundary-1}:\quad \alpha_0=\infty,\quad \alpha_1=\infty \tag{2.3a}$$

$$x\text{-boundary-2}:\quad \alpha_0=\infty,\quad \alpha_1=0 \tag{2.3b}$$

$$x\text{-boundary-3}:\quad \alpha_0=0,\quad \alpha_1=0 \tag{2.3c}$$

$$x\text{-boundary-4}:\quad \alpha_0=0,\quad \alpha_1=\infty \tag{2.3d}$$

Cases with x-boundary-2 and x-boundary-4 are the same mathematically. With x-boundary-3, the solution can be approximated as one-dimensional solution in r-direction which has been studied earlier. Therefore, we shall only consider two boundary conditions: x-boundary-1 and x-boundary-2. And closed form solutions will be provided.

3. Solution method

3.1. Homogenising the equations

With x-boundary-1: $\alpha_0=\infty$, $\alpha_1=\infty$, boundary conditions in Eqs. (2.1f–g) are

$$T_j(t,r,0)=T_\infty(t),\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{3.1a}$$

$$T_j(t,r,1)=T_\infty(t),\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{3.1b}$$

For any jth layer, we introduce the following new variable in order to homogenise some of the boundaries

$$U_j=T_j-T_\infty(t) \tag{3.2}$$

This leads to the following equation for Eq. (2.1)

$$k_j\left(\frac{\partial^2U_j}{\partial r^2}+\frac{1}{r}\frac{\partial U_j}{\partial r}\right)+k_j\frac{\partial^2U_j}{\partial x^2}=\frac{\partial U_j}{\partial t}+T'_\infty(t),\quad r\in[r_{j-1},r_j],\ \ x\in[0,1],\ \ j=1,\dots,n \tag{3.3a}$$

with boundaries

$$-\lambda_1\frac{\partial U_1}{\partial r}(t,r_0,x)=-\alpha_+(U_1(t,r_0,x)+T_\infty(t)-T_+(t)),\quad x\in[0,1] \tag{3.3b}$$

$$U_j(t,r_j,x)=U_{j+1}(t,r_j,x),\quad x\in[0,1],\ \ j=1,\dots,n-1 \tag{3.3c}$$

$$-\lambda_j\frac{\partial U_j}{\partial r}(t,r_j,x)=-\lambda_{j+1}\frac{\partial U_{j+1}}{\partial r}(t,r_j,x),\quad x\in[0,1],\ \ j=1,\dots,n-1 \tag{3.3d}$$

$$-\lambda_n\frac{\partial U_n}{\partial r}(t,r_n,x)=\alpha_\infty U_n(t,r_n,x),\quad x\in[0,1] \tag{3.3e}$$

$$U_j(t,r,0)=0,\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{3.3f}$$

$$U_j(t,r,1)=0,\quad r\in[r_{j-1},r_j],\ \ j=1,\dots,n \tag{3.3g}$$

$$U_j(0,r,x)=-T_\infty(0),\quad r\in[r_{j-1},r_j],\ \ x\in[0,1],\ \ j=1,\dots,n \tag{3.3h}$$

3.2. Separating the variables

Traditionally, the application of 'separation of variables' needs that the equations be linear and homogeneous. Unfortunately, this is not true in our targeted equations. Therefore, we adopted a novel technique which is different from those commonly reported papers [8].

Assume 'separation of variables' can be used as

$$U_j(t,r,x)=u_j(t,r)X_j(x) \tag{3.4}$$

where $X_j(x)$ is a variable-separated function which satisfies the homogeneous form of Eq. (3.3a). Then by substituting $X_j(x)$ into Eq. (3.3a) results in

$$\text{function of } t \quad \text{and} \quad r = \frac{k_j X_j''}{X_j} \tag{3.5}$$

Setting each side of the above equation equal $-\mu_j^2$ gives

$$X_j'' + \frac{\mu_j^2}{k_j} X_j = 0 \tag{3.6a}$$

The general solution of (3.6a) is then obtained as

$$X_{jm} = A_{jm} \sin\left(\frac{\mu_{jm}}{\sqrt{k_j}} x\right) \tag{3.6b}$$

Combining the boundaries (3.3f–g), $X_{jm}(0) = 0$ and $X_{jm}(1) = 0$, leads to

$$\frac{\mu_{jm}}{\sqrt{k_j}} = m\pi \quad \text{or} \quad \mu_{jm} = m\pi\sqrt{k_j} \quad \text{and}$$
$$X_{jm}(x) = X_m(x) = \sin(m\pi x), \quad m = 1, \ldots, \infty \tag{3.7}$$

Solution U_j in Eq. (3.4) can then be expressed as

$$U_j(t, r, x) = \sum_{m=1}^{\infty} u_{jm}(t, r) X_m(x) = \sum_{m=1}^{\infty} u_{jm}(t, r) \sin(m\pi x) \tag{3.8}$$

Note that the coefficient A_{jm} in Eq. (3.6b) is embedded in u_{jm} in Eq. (3.8).

3.3. Resultant one-dimensional heat equation in t and x variables

We shall omit writing $m = 1, \ldots, \infty, j = 1, \ldots, n$ etc. if it cannot cause confusion. Note that $X_m(x) = \sin(m\pi x)$ are orthogonal functions. Representing 1 as a sum of $X_m(x)$ and combining Eq. (3.8), Eq. (3.3a) is then rewritten as

$$k_j\left(\sum_{m=1}^{\infty} \frac{\partial^2 u_{jm}}{\partial r^2} X_m + \frac{1}{r}\sum_{m=1}^{\infty} \frac{\partial u_{jm}}{\partial r} X_m\right) - \sum_{m=1}^{\infty} \mu_{jm}^2 u_{jm} X_m = \sum_{m=1}^{\infty} \frac{\partial u_{jm}}{\partial t} X_m + T_\infty'(t) \sum_{m=1}^{\infty} b_m X_m \tag{3.9a}$$

where

$$b_m = \frac{2(1 - \cos(m\pi))}{m\pi} \tag{3.9b}$$

Similarly, the same trick can be applied to the boundary conditions (3.3b–h). Finally, we get to the following equation system:

$$k_j\left(\frac{\partial^2 u_{jm}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{jm}}{\partial r}\right) - \mu_{jm}^2 u_{jm} = \frac{\partial u_{jm}}{\partial t} + b_m T_\infty'(t) \quad r \in [r_{j-1}, r_j], \; j = 1, \ldots, n \tag{3.10a}$$

with boundaries

$$-\lambda_1 \frac{\partial u_{1m}}{\partial r}(t, r_0) = -\alpha_+(u_{1m}(t, r_0) + b_m T_\infty(t) - b_m T_+(t)) \tag{3.10b}$$

$$u_{jm}(t, r_j) = u_{(j+1)m}(t, r_j), \quad j = 1, \ldots, n-1 \tag{3.10c}$$

$$-\lambda_j \frac{\partial u_{jm}}{\partial r}(t, r_j) = -\lambda_{j+1} \frac{\partial u_{(j+1)m}}{\partial r}(t, r_j) \quad j = 1, \ldots, n-1 \tag{3.10d}$$

$$-\lambda_n \frac{\partial u_{nm}}{\partial r}(t, r_n) = \alpha_\infty u_{nm}(t, r_n) \tag{3.10e}$$

$$u_{jm}(0, r) = -b_m T_\infty(0) \quad r \in [r_{j-1}, r_j], \; j = 1, \ldots, n \tag{3.10f}$$

Note that Eq. (3.10f) is obtained by expressing 1 as a sum of X_m in Eq. (3.3h). It can be observed that the derived one-dimensional transient heat equation for r-variable exhibits quite different mathematical form than the original multi-dimensional equation does.

3.4. Closed form solution

For any jth layer, we introduce the following new variable as

$$R_{jm} = u_{jm} + \frac{i\omega_\infty b_m}{\mu_{jm}^2 + i\omega_\infty} T_\infty(t) + \frac{\mu_{jm}^2 b_m}{\mu_{jm}^2 + i\omega_\infty} e^{-\mu_{jm}^2 t + i\varphi} \tag{3.11}$$

Eq. (3.10a) is then homogenised as

$$k_j\left(\frac{\partial^2 R_{jm}}{\partial r^2} + \frac{1}{r}\frac{\partial R_{jm}}{\partial r}\right) - \mu_{jm}^2 R_{jm} = \frac{\partial R_{jm}}{\partial t} \tag{3.12a}$$

with boundaries

$$-\lambda_1 \frac{\partial R_{1m}}{\partial r}(t, r_0) = -\alpha_+(R_{1m}(t, r_0) - R_{+m}(t)) \tag{3.12b}$$

$$R_{jm}(t, r_j) = R_{(j+1)m}(t, r_j), \quad j = 1, \ldots, n-1 \tag{3.12c}$$

$$-\lambda_j \frac{\partial R_{jm}}{\partial r}(t, r_j) = -\lambda_{j+1} \frac{\partial R_{(j+1)m}}{\partial r}(t, r_j), \quad j = 1, \ldots, n-1 \tag{3.12d}$$

$$-\lambda_n \frac{\partial R_{nm}}{\partial r}(t, r_n) = -\alpha_\infty(R_{-m}(t) - R_{nm}(t, r_n)) \tag{3.12e}$$

$$u_{jm}(0, r) = 0, \tag{3.12f}$$

where

$$T_+(t) = e^{i\omega_+ t + i\varphi_+} \tag{3.12g}$$

$$T_\infty(t) = e^{i\omega_\infty t + i\varphi_\infty} \tag{3.12h}$$

$$R_{+m}(t) = -\frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega_\infty} T_\infty(t) + \frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + i\omega_\infty} \mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi} + b_m T_+(t) \tag{3.12i}$$

$$R_{-m}(t) = \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega_\infty} T_\infty(t) + \frac{\mu_{nm}^2 b_m}{\mu_{nm}^2 + \mathrm{i}\omega_\infty} \mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi} \tag{3.12j}$$

Eq. (3.12) presents a similar system of transient one-dimensional heat equation in composite hollow cylinder which has been studied by authors as an extension of [9], where $R_{+m}(t)$ and $R_{-m}(t)$ represented boundary temperatures and the convective term $-\mu_{jm}^2 R_{jm}$ was missing. However, application of Laplace transform on equation will result in exactly same type of ordinary equation. Hence, without showing the details, we give the closed form solution for R_{jm} as following:

For jth layer, denote

$$\overline{R_{jm}}(s) = \int_0^\infty e^{-s\tau} R_{jm}(\tau)\,\mathrm{d}\tau,$$

$$q_j = \sqrt{\frac{s+\mu_{jm}^2}{k_j}} = \sqrt{s + m^2\pi^2},$$

$$h_0 = \frac{\alpha_+}{\lambda_1 q_1}, \quad h_j = \frac{\lambda_{j+1}}{\lambda_j}\sqrt{\frac{k_j}{k_{j+1}}}, \; j = 1,\ldots,n-1, \tag{3.13}$$

$$h_n = -\frac{\alpha_\infty}{\lambda_n q_n}, \quad \eta_j = q_j r_{j-1},$$

$$\xi_j = q_j r_j, \; j = 1,\ldots,n$$

$$\Delta(s) = \begin{vmatrix} I_1(\eta_1) - h_0 I_0(\eta_1) & -K_1(\eta_1) - h_0 K_0(\eta_1) & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ I_0(\xi_1) & K_0(\xi_1) & -I_0(\eta_2) & -K_0(\eta_2) & \ldots & 0 & 0 & 0 & 0 \\ I_1(\xi_1) & -K_1(\xi_1) & -h_1 I_1(\eta_2) & h_1 K_1(\eta_2) & \ldots & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & I_0(\xi_{n-1}) & K_0(\xi_{n-1}) & -I_0(\eta_n) & -K_0(\eta_n) \\ 0 & 0 & 0 & 0 & \ldots & I_1(\xi_{n-1}) & -K_1(\xi_{n-1}) & -h_{n-1} I_1(\eta_n) & h_{n-1} K_1(\eta_n) \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & I_1(\xi_n) - h_n I_0(\xi_n) & -K_1(\xi_n) - h_n K_0(\xi_n) \end{vmatrix} \tag{3.14a}$$

$$\Delta_1(s) = -h_0 \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \quad \Delta_2(s) = h_n \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j - 1 \\ \text{deleted} & \end{vmatrix}}{\Delta(s)} \tag{3.14b}$$

$$\Delta_3(s) = h_0 \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 1 & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)}, \quad \Delta_4(s) = -h_n \frac{\begin{vmatrix} \Delta(s) & \text{with} \\ \text{row} - 2n & \text{column} - 2j \\ \text{deleted} & \end{vmatrix}}{\Delta(s)} \tag{3.14c}$$

$$F_m(s,r) = [\Delta_1 I_o(q_j r) + \Delta_3 K_o(q_j r)], \quad G_m(s,r) = [\Delta_2 I_o(q_j r) + \Delta_4 K_o(q_j r)] \tag{3.15a}$$

$$R_{jm} = \mathrm{real}\Bigg(-\frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega_\infty} F_m(\mathrm{i}\omega_\infty, r) T_\infty(t) + \frac{\mu_{1m}^2 b_m}{\mu_{1m}^2 + \mathrm{i}\omega_\infty} F_m(-\mu_{1m}^2, r)\mathrm{e}^{-\mu_{1m}^2 t + \mathrm{i}\varphi} + b_m F_m(\mathrm{i}\omega_+, r) T_+(t) + \frac{\mathrm{i}\omega b_m}{\mu_{nm}^2 + \mathrm{i}\omega_\infty} G_m(\mathrm{i}\omega_\infty, r) T_\infty(t) + \frac{\mu_{nm}^2 b_m}{\mu_{nm}^2 + \mathrm{i}\omega_\infty} G_m(-\mu_{nm}^2, r)\mathrm{e}^{-\mu_{nm}^2 t + \mathrm{i}\varphi} \Bigg) \tag{3.15b}$$

where real represents the real part of the function.The combination of Eqs. (3.2), (3.8), (3.11) and (3.15b) gives the final closed form solution as

$$T_j = \mathrm{real}\left[\sum_{m=1}^{\infty} \left(R_{jm} - \frac{\mathrm{i}\omega_\infty b_m}{\mu_{jm}^2 + \mathrm{i}\omega_\infty} T_\infty(t) - \frac{\mu_{jm}^2 b_m}{\mu_{jm}^2 + \mathrm{i}\omega_\infty} \mathrm{e}^{-\mu_{jm}^2 t + \mathrm{i}\varphi} \right) \times \sin(m\pi x) + T_\infty(t) \right] \tag{3.16}$$

where R_{jm} is given in Eq. (3.15b).

3.5. Solution to the second x-boundary condition

The second x-boundary condition requires that $\alpha_0 = \infty$, $\alpha_1 = 0$. Then the boundary condition (2.1f–g) becomes (see equations)

$$T_j(t,r,0) = T_\infty(t), \quad r \in [r_{j-1}, r_j], \; j = 1,\ldots,n \tag{3.17a}$$

$$\frac{\partial T_j}{\partial x}(t,r,1) = 0, \quad r \in [r_{j-1}, r_j], \; j = 1,\ldots,n \tag{3.17b}$$

Eq. (3.3) keeps the same except (3.3g) which becomes

$$\frac{\partial U_j}{\partial x}(t,r,1) = 0 \tag{3.18}$$

Separating variables results in Eqs. (3.5) and (3.6). Boundary condition (3.17a–b) requires that $X_j(0) = 0$ and $\frac{\partial X_j}{\partial x}(1) = 0$. Therefore, Eq. (3.7) is changed as

$$\frac{\mu_j}{\sqrt{k_j}} = \left(m+\frac{1}{2}\right)\pi \quad \text{or} \quad \mu_{jm} = \left(m+\frac{1}{2}\right)\pi\sqrt{k_j}$$

and

$$X_{jm}(x) = X_m(x) = \sin\left(\left(m+\frac{1}{2}\right)\pi x\right), \quad m = 1,\ldots,\infty \tag{3.19}$$

Using the orthogonal property of X_m to express 1 as a sum of X_m, we get exactly the same heat conduction equation as Eq. (3.9) except that μ_{jm} and b_m are given in Eq. (3.19) and the following, respectively:

$$b_m = \frac{2(1-\cos(m\pi+\frac{\pi}{2}))}{(2m+1)\pi}, \quad m = 1,\ldots,\infty \tag{3.20}$$

The solution procedure follows exactly the same methodology developed for the first x-boundary condition.

3.6. Solution to more generally time-dependent boundary condition

For more generally time-dependent boundary conditions, present the boundaries as Fourier series as $T_+(t) = a_{+0} + \sum_{k=1}^{\infty} a_{+k}\cos(\omega_{+k}t + \varphi_{+k})$, $T_\infty(t) = a_{\infty 0} + \sum_{k=1}^{\infty} a_{\infty k}\cos(\omega_{\infty k}t + \varphi_{\infty k})$. By linear property, the corresponding solution can be expressed as the sum of solutions with constant boundary temperatures and with infinitely sums of cosines. Solution to the constant boundary temperature is approximated by linearisation of hyperbolic functions sinh and cosh in F, G in Eq. (3.15a) to obtain $\overline{R_{jm}}(s) \approx \frac{\text{const}}{\text{const1}*s+\text{const2}}$. Hence $R_{jm}(t) = \frac{\text{const}}{\text{const1}}\exp(-\frac{\text{const2}}{\text{const1}}t)$. If studies do not focus very much on the initial temperature change, the steady state solution may also be a good approximation to the constant boundary temperature. The solution to infinitely sums of cosines is easily obtained from the previous theory due to the linear property.

4. Calculation example

A five-layer composite circular cylinder was selected as the calculation example demonstrated in Fig. 2. Thermal properties and dimensions of the slab are given in Table 1. Surface heat transfer coefficients were set as $\alpha_\infty = 25\ \text{W/m}^2/\text{K}$ and $\alpha_+ = 6\ \text{W/m}^2/\text{K}$.

The boundary temperatures were taken from measurements and fitted with periodic functions with periods 30, 5, 2 and 1 days as

$$T_+(t) = a_{+0} + \sum_{i=1}^{4} a_{+i}\cos\left(\frac{2\pi t}{\omega_i} - \varphi_{+i}\right) \tag{4.1a}$$

$$T_\infty(t) = a_{\infty 0} + \sum_{i=1}^{4} a_{\infty i}\cos\left(\frac{2\pi t}{\omega_i} - \varphi_{\infty i}\right) \tag{4.1b}$$

where fitting parameters are listed in Table 2 and Fig. 3 shows the values.

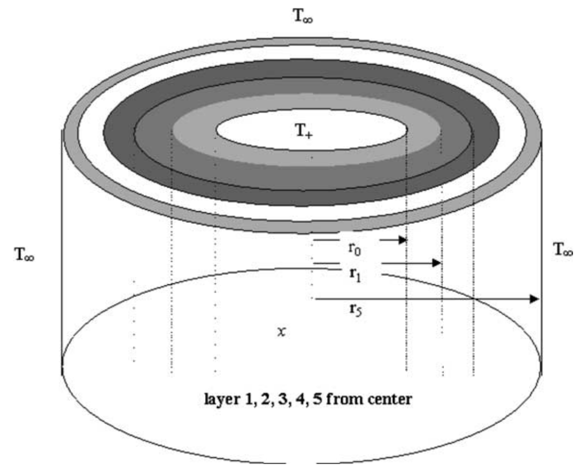

Fig. 2. Schematic picture of the five-layer circular cylinder calculated in the example.

Table 1
Material properties and dimensions of the five-layer circular cylinder

Layer	Thermal conductivity (W/m/K)	Thermal diffusivity (m^2/s)	Thickness (mm)
1	0.23	4.11×10^{-7}	50
2	0.0337	1.47×10^{-6}	100
3	0.9	3.75×10^{-7}	100
4	0.147	1.61×10^{-7}	200
5	0.12	1.5×10^{-7}	20

Table 2
Parameters of Eq. (4.1)

	ω_1	ω_2	ω_3	ω_4
	30.0	5.0	2.0	1.0
	φ_{+1}	φ_{+2}	φ_{+3}	φ_{+4}
	5.149231	16.77994	−0.67884	4.381328
a_{+0}	a_{+1}	a_{+2}	a_{+3}	a_{+4}
17.0	1.919486	0.732953	−0.25824	0.132831
	$\varphi_{\infty 1}$	$\varphi_{\infty 2}$	$\varphi_{\infty 3}$	$\varphi_{\infty 4}$
	5.607506	13.59596	1.451539	5.418717
$a_{\infty 0}$	$a_{\infty 1}$	$a_{\infty 2}$	$a_{\infty 3}$	$a_{\infty 4}$
5.0	2.72217	−5.019664	1.084058	0.4648

Calculated points were made in the central points of each layer (e.g. represented as layer 3 to layer 4 in the figure). Fig. 4 displays the comparison of the transient temperature variation using the analytical and numerical methods. The temperatures were stored in files as hourly values and shown in figures as hourly and daily values. In the figure, boundary conditions are presented as $T_+(t) - T_\infty(t)$ (represented as $T_+ - T_{\text{inf}}$ in the figure) for convenience. The corresponding temperatures in lay-

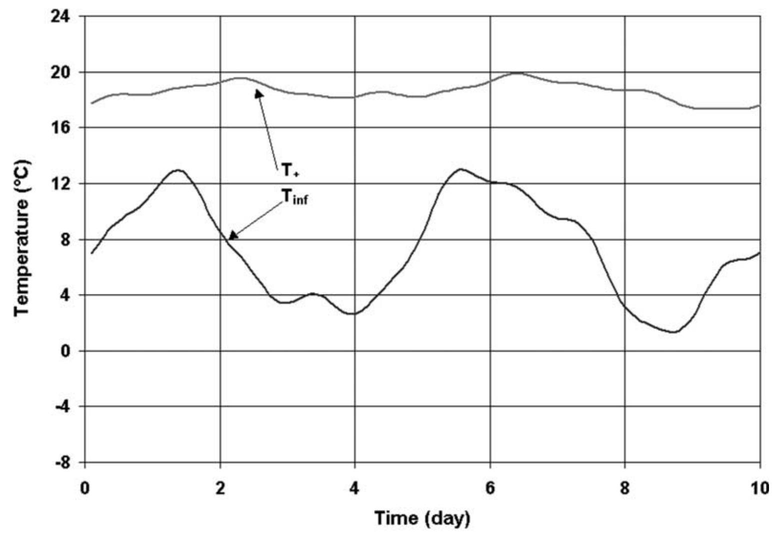

Fig. 3. Time-dependent boundary condition.

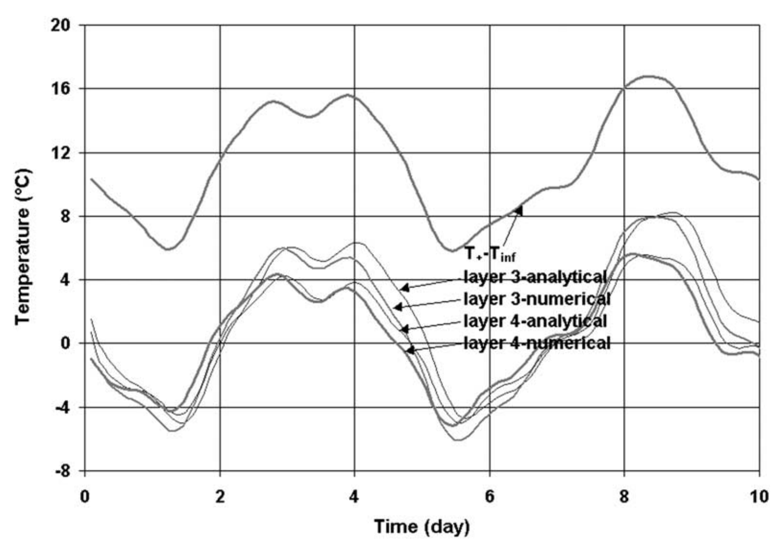

Fig. 4. Comparison of analytical and numerical results, layers three and four.

ers are plotted as $T_j(t) - T_\infty(t)$. It can be seen that the numerical agrees with analytical results. The results for the first two days are provided in Fig. 5. More calculation results in other layers did not show any substantial change. Therefore, we only illustrate the results in layer three and four.

5. Discussions

The solution of transient multi-dimensional heat conduction of n-layer circular cylinder is explicitly expressed through Eq. (3.16). We make some observations.

- Calculation includes only simple computation of matrix determinant which can be easily accomplished by commercial mathematical packages like Maple, Matlab and Mathematica and even by hands. No numerical work is necessitated. For any jth layer, only five sparse matrices are involved. The calculation load is small and the computing time is short.
- Compared with numerical methods, the developed method is easier to complement and a possible instability in numerical method is avoided. This is especially important for multi-dimensional heat conduction problem as imaginary eigenvalues may exist which cause instability of the numerical program.

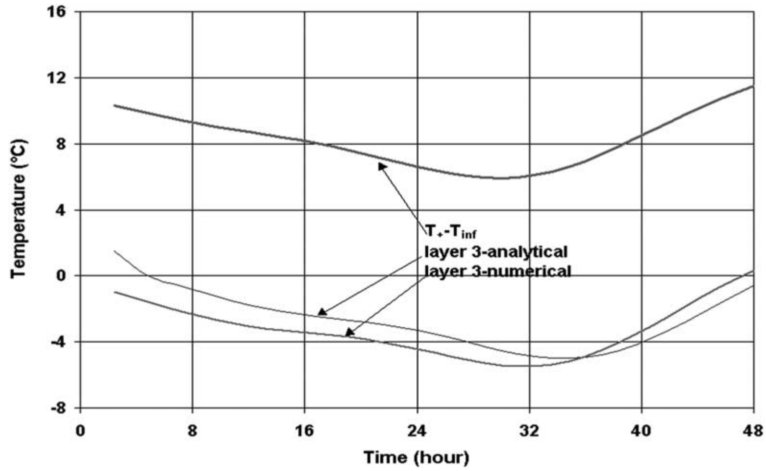

Fig. 5. Comparison of analytical and numerical results, layer three.

- It is known that any periodic and piecewise continuous function can be approximated as its Fourier expansion. Hence, time-dependent boundary temperature can be approximated by Fourier series. Therefore, boundary conditions are not restricted to periodic ones as demonstrated in the calculation example.
- It is only for the demonstration sake that the assumptions of constant conductivity axially and radially in each layer and perfect contact between layers are presumed. However, observing the analytical technique we developed, these two restrictions can be easily taken away without adding contents in the paper.

6. Conclusions

In this paper, an analytical approach to multi-dimensional heat conduction in composite circular cylinder subject to generally time-dependent temperature changes has been presented. Boundary temperatures were approximated as Fourier series. Laplace transform was adopted in deducing the solutions. The solution was approximated without evaluating the residues. The application of 'separation of variables' was novel in multi-dimensional case which led to an almost same amount of calculation load as in one-dimensional problem. An n-layer closed form solution is provided which is lacking in literatures. The method is shown to have considerable potential in solving heat conduction equation. These conclusions have also been demonstrated in solving multi-dimensional heat conduction in composite cylinder in our companion paper.

Acknowledgement

This research was supported by the Academy of Finland.

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